Mock exam multivariable analysis Jan 2020

Exercise 1

a. Prove that the intersection of two linear subspaces of dimension $(n-1)$ in $\mathbb{R}^n$ cannot be a single point when $n > 2$.

b. Suppose that for a C^1-differentiable $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^2$ we have $f^{-1}(\{0\}) = \{e_1\}$. Can $f'(e_1)$ have a two-dimensional image?

c. Prove that the system of ordinary differential equations $x'(t) = y(t)x(t)$ and $y'(t) = x(t)+2y(t)$ for real valued functions x, y has a solution such that $x(0) = y(0) = 10$.

Exercise 2

a. Prove that if $f, g : P \to P$ are two contractions on $P \subset \mathbb{R}^n$ then so is their composition $g \circ f$.

b. Suppose $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$ is a C^1 function such that the directional derivatives $\partial_w(0)$ are zero for all vectors $w \in \mathbb{R}^n$ such that $e^1(w) = 29$. Show that $f'(0) = 0$.

c. Prove that if $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$ is differentiable and satisfies $f(0) = 0$ then $\frac{|f'(0)h - f(h)|}{|h|}$ converges to 0 as h converges to 0.

Exercise 3

a. Find two different 2-cubes γ, δ in $\mathbb{R}^4$ with equal boundary.

b. Prove or give a counter example: For any γ, δ as above and all $\omega \in \Omega^2(\mathbb{R}^4)$ we have $\int_\gamma \omega = \int_\delta \omega$.

c. Suppose $S \subset \mathbb{R}^n$ is star-shaped and $T \subset \mathbb{R}^n$ is not star-shaped. Imagine a C^1 map $T \xrightarrow{f} S$ and $\omega \in \Omega^k(S)$ such that $d\omega = 0$, prove that there exists an $\alpha \in \Omega^{k-1}(T)$ such that $d\alpha = f^*\omega$

Exercise 4

a. Compute $\int_{\partial I^k} \omega$, where $\omega \in \Omega^{k-1}(\mathbb{R}^k)$ is defined by $x^k dx^1 \wedge \cdots \wedge dx^{k-1}$.

b. Compute $F^*\omega$ where F is a C^1 function on $[0,1]^k$ with values in $[0,1]^k \times \{0\}$.

c. Show there cannot be a C^1 function $[0,1]^k \xrightarrow{F} [0,1]^k$ such that $\partial I^k = \partial F$ and the image of F is in $[0,1]^k \times \{0\}$. hint: Compare $\int_{\partial I^k} \omega$ to $\int_{\partial I^k} F^*\omega$.

Solutions

Exercise 1

a. Prove that the intersection of two linear subspaces of dimension $(n-1)$ in $\mathbb{R}^n$ cannot be a single point, when $n > 2$.
If the intersection is a single point that must be $\{0\}$. Choose a basis $a_1, \dots a_{n-1}$ and $b_1, \dots b_{n-1}$ for each of the two subspaces A and B. Any linear relation between the a's and the b's must involve at least one a and at least one b because the a's and the b's are bases of A and B. But such a linear relation would then allow us to express a combination of a's in terms of a combination of b's which means those combinations are a non-zero vector in the intersection $A \cap B = \{0\}$. Therefore no such relation exists and we found $2n-2$ independent vectors in $\mathbb{R}^n$, which is absurd if $n > 2$.

b. Suppose that for a C^1-differentiable $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^2$ we have $f^{-1}(\{0\}) = \{e_1\}$. Can $f'(e_1)$ have a two-dimensional image?

If the image is two-dimensional then there must exist $i \neq j$ such that $f'(e_1)e_i$ and $f'(e_1)e_j$ span $\mathbb{R}^2$. After permuting the coordinates we may assume that $i = n-1$ and $j = n$. Then we may apply the implicit function theorem to the map $\mathbb{R}^{n-2} \times \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$ with $f(x_0, y_0) = z_0$ and $(x_0, y_0) = (1, 0, 0, \dots, 0)$ and $z_0 = (0, 0)$. The condition that $F(y) = f(x_0, y)$ has invertible derivative $F'(0)$ is satisfied because $F'(0)e_1 = f'(e_1)e_{n-1}$ and $F'(0)e_2 = f'(e_1)e_n$ span $\mathbb{R}^2$. The conclusion is that there is an open subset $N \subset \mathbb{R}^{n-2}$ containing e_1 with a C^1 function $N \xrightarrow{g} M$ such that $f_{-1}(\{0\})$ is locally the graph of g near point e_1. This contradicts our assumption that $f^{-1}(\{0\})$ is a single point.

c. Prove that the system of ordinary differential equations $x'(t) = y(t)x(t)$ and $y'(t) = x(t)+2y(t)$ for real valued functions x, y has a solution such that $x(0) = y(0) = 10$.
Define the vector field $\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2$ by $F(x, y) = (yx, x + 2y)$. This is clearly a C^1 function because the partial derivatives exist and are polynomials, hence continuous. The existence and uniqueness theorem for ordinary differential equations (Picard) says that there exists and integral curve $(-a, a) \xrightarrow{\gamma} \mathbb{R}^2$ with $\gamma(0) = (10, 10)$. This means that $\gamma'(t) = F(\gamma(t))$ for all $t \in (-a, a)$ and writing $\gamma(t) = (x(t), y(t))$ we see that γ is the desired solution to the differential equation.

Exercise 2

a. Prove that if $f, g : P \to P$ are two contractions on $P \subset \mathbb{R}^n$ then so is their composition $g \circ f$.
There exist $\alpha, \beta \in (0, 1)$ such that for all $x, y \in P$ we have $|f(x) - f(y)| < \alpha|x - y|$ and $|g(x) - g(y)| < \beta|x - y|$. Therefore $|g(f(x)) - g(f(y))| < \beta|f(x) - f(y)| < \alpha\beta|x - y|$.

b. Suppose $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$ is a C^1 function such that the directional derivatives $\partial_w(0)$ are zero for all vectors $w \in \mathbb{R}^n$ such that $e^1(w) = 29$. Show that $f'(0) = 0$.
We have $\partial_w(0) = f'(0)w$ by the chain rule applied to the function $t \mapsto t + w$ composed with f. The set of vectors w with $e^1(w) = 29$ spans $\mathbb{R}^n$ so $f'(0)w = 0$ for all these vectors implies $f'(0)v = 0$ for all vectors because $f'(0)$ is linear.

c. Prove that if $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$ is differentiable and satisfies $f(0) = 0$ then $\frac{|f'(0)h - f(h)|}{|h|}$ converges to 0 as h converges to 0.
By definition of the derivative $f'(0)$ we have $\lim_{h \to 0} \frac{|f(0+h)-f(0)-f'(0)h|}{|h|} = 0$ since that is equivalent to the error $\epsilon_{f,0}(h)$ being $o(h)$.

Exercise 3

a. Find two different 2-cubes γ, δ in $\mathbb{R}^4$ with equal boundary.
Take $\gamma(s,t) = (s,t,0,0)$ and $\delta(s,t) = (s,t,st(s-1)(t-1),0)$. Since $st(s-1)(t-1)$ is 0 whenever either s or t is in $\{0,1\}$ we have $\gamma_{i,\sigma} = \delta_{i,\sigma}$. This implies $\partial\gamma = \partial\delta$.

b. Prove or give a counter example: For any γ, δ as above and all $\omega \in \Omega^2(\mathbb{R}^4)$ we have $\int_\gamma \omega = \int_\delta \omega$. This is not true. Choose δ and γ as above and $\omega = zdx \wedge dy$. Then $\delta'(s,t)e_1 = (1,0,t(t-1)(2s-1))$ and $\delta'(s,t)e_2 = (1,0,(2t-1)s(s-1))$. Therefore $\delta'(s,t)^*e^1 = e^1$ and $\delta'(s,t)^*e^2 = e^2$, in other words $\delta'(s,t)^*dx = ds$ and $\delta'(s,t)^*dy = dt$. Therefore $\delta^*\omega(s,t) = \delta'(s,t)^*\omega = st(s-1)(t-1)ds \wedge dt$ and so $\int_\delta \omega = \int_{I^2} \delta^*\omega = \int_{(s,t)\in[0,1]^2} st(s-1)(t-1) = 1/36$. The other integral $\int_\gamma \omega$ is zero since $e^3(\gamma(s,t)) = 0$ for all s,t, which means $\gamma^*\omega = 0$ as well.

c. Suppose $S \subset \mathbb{R}^n$ is star-shaped and $T \subset \mathbb{R}^n$ is not star-shaped. Imagine a C^1 map $T \xrightarrow{f} S$ and $\omega \in \Omega^k(S)$ such that $d\omega = 0$, prove that there exists an $\alpha \in \Omega^{k-1}(T)$ such that $d\alpha = f^*\omega$
By the Poincaré lemma we find a $k-1$-form η on S such that $d\eta = \omega$. The pull-back $\alpha = f^*\eta$ is a $k-1$-form on T such that $d\alpha = df^*\eta = f^*d\eta = f^*\omega$.

Exercise 4

a. Compute $\int_{\partial I^k} \omega$, where $\omega \in \Omega^{k-1}(\mathbb{R}^k)$ is defined by $x^k dx^1 \wedge \cdots \wedge dx^{k-1}$.
$d\omega = (-1)^{k-1} e^{(1,2,\ldots,k)}$ so by Stokes $\int_{\partial I^k} \omega = \int_{I^k} d\omega = (-1)^{k-1}$.

b. Compute $F^*\omega$ where F is a C^1 function on $[0,1]^k$ with values in $[0,1]^k \times \{0\}$.
$F^*\omega(p) = F'(p)^*\omega(F(p)) = 0$ because $\omega(F(p)) = 0$ for all $p \in [0,1]^k$ as we should set the k-th coordinate to 0.

c. Show there cannot be a C^1 function $[0,1]^k \xrightarrow{F} [0,1]^k$ such that $\partial I^k = \partial F$ and the image of F is in $[0,1]^k \times \{0\}$. hint: Compare $\int_{\partial I^k} \omega$ to $\int_{\partial I^k} F^*\omega$.
If there were such a function then $0 \neq \int_{\partial I^k} \omega = \int_{\partial F} \omega = \int_F d\omega = \int_{I^k} F^*d\omega = \int_{I^k} dF^*\omega = \int_{\partial I^k} F^*\omega = 0$.